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LETTERS TO THE EDITOR

EFFECTS OF A DELAMINATION AND INITIAL CURVATURE ON THE FUNDAMENTAL FREQUENCY OF SANDWICH BEAMS

J. N. ROSSETTOS

Department of Mechanical Engineering, Northeastern University, Boston, MA 02115-5096, *U.S.A.*

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1. INTRODUCTION

In many structural applications, especially those in aerospace, curved panels are frequently encountered. Sandwich panels, in particular, where thin face sheets are separated by a thick layer (core) of weak but very lightweight material, also provide a means to increase the flexural stiffness for a given weight. The effects of panel curvature in the vibration of homogeneous beams were treated earlier in reference [1] and references [2, 3, 4]. For slight curvature, where the initial shape of the beam centerline is quadratic, a closed form solution was obtained in reference [3] for the vibration of curved Timoshenko beams. It was shown that even very slight curvature tends to increase the fundamental frequency sharply for simply supported and clamped beams. Higher modes were less sensitive to slight curvature, and in general, for cases where shear deformation becomes more important, initial curvature had a correspondingly smaller influence on the results.

In the case of curved sandwich beams, the finite element method was used in reference [5] to study the effects of various geometric parameters on the resonant frequency, emphasis being given to calculation of loss factors. Curvature was shown to make a difference in the first three modes, becoming less important in higher modes. In reference [6], some theoretical results of the resonant frequency are verified experimentally where the energy method was used to derive the governing equations for the transverse vibration of sandwich beams, and a series solution applicable to simply supported beams was assumed. The Ritz method was used in reference [7] to perform a parametric study showing the effects of curvature, core thickness and adhesive shear modulus on the resonant frequency and loss factors. It is also mentioned that in the case of straight sandwich beams, natural frequencies were obtained in reference [8] using the Lagrange Multiplier method and showed good agreement between theory and experiment.

In the present work, a slightly curved sandwich beam, which may contain a delamination is considered. A non-dimensional variational statement is developed using the principle of minimum potential energy, where the strain energy is based on the well-known model by Hoff [9, 10] for straight sandwich beams, and applied to sandwich plates by Thurston [11]. Here, the sandwich beam also includes the strain energy due to initial curvature. The important parameters which result involve initial curvature (or initial beam rise), core shear deformation, core/face sheet geometry and delamination length. The present study exhibits the interesting interplay of initial curvature, transverse shear deformation and delamination length for the first time. For instance, it is shown that when initial curvature increases, the influence of shear deformation becomes less. Also, the sensitivity to the length of delamination damage is greater for straight beams than for initially curved beams. It is also greater for cases with less shear deformation (i.e., high modulus cores).

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2. ANALYSIS

It is assumed that the two equal face sheets take on bending and axial strain energy while the core takes only shear energy. The strain energy can be expressed in terms of the extensional displacement, u, and the bending displacement, v, of the face sheets. The transverse displacement, v, does not vary through the thickness since the normal strain in the core is considered negligible. Note (see Figure 1) that u is the displacement in the positive x direction of the centerline of the upper face and an equal opposite displacement in the lower face. Then, according to the Hoff model, which now includes initial curvature, the strain energy can be written as

$$U_s = \frac{1}{2}\int_0^L [2E_f I_f v_{,xx}^2 + 2E_f A_f (u_{,x} - v_{o,xx}v)^2 + G_c A_c (2u/h - v_{,x})^2]\, \mathrm{d}x, \tag{1}$$

where $I_f = bt^3/12$, $A_f = bt$, $A_c = bc$, $h = c + t$, E_f is the Young's modulus of the face sheets and G_c is the shear modulus of the core. The third term in the integrand of equation (1) gives the energy due to transverse shear deformation in the core. The second term of the integrand contains the effect of initial beam curvature, $v_{o,xx}$, on the axial strain in the face sheets as used in references [3, 4] for curved homogeneous beams. The term, v_o, is the initial shape of the beam centerline and $()_{,x} = \mathrm{d}()/\mathrm{d}x$, etc. The subscripts f and c denote face sheets and core respectively.

The corresponding kinetic energy is

$$T = \frac{1}{2}\int_0^L [(2m_f + m_c)\dot{v}^2 + 2m_f \dot{u}^2 + I_{mc}(2\dot{u}/h)^2]\, \mathrm{d}x, \tag{2}$$

where m_f and m_c are mass per unit length for each face and core, I_{mc} is mass moment of inertia of the core per unit length, and dots denote time differentiation. If the beam is undergoing simple harmonic motion of frequency ω so that $(u, v) = (u, v)\sin \omega t$, the potential energy due to inertial loading, based on equation (2) can be written as (see also reference [12])

$$U_\omega = -\frac{1}{2}\int_0^L [(2m_f + m_c)\omega^2 v^2 + 2m_f \omega^2 u^2 + I_{mc}\omega^2 (2u/h)^2]\, \mathrm{d}x. \tag{3}$$

The principle of minimum potential energy then states that

$$\delta L_r = 0, \qquad \text{where} \qquad L_r = U_s + U_\omega. \tag{4}$$

It is convenient to define the non-dimensional quantities V, U and ξ as

$$v = hV, \qquad u = (h^2/L)U, \qquad x = L\xi, \qquad \bar{e} = Le, \tag{5}$$

where L is the length of the beam. In terms of the non-dimensional co-ordinate in the axial direction, ξ, the initial shape of the beam is taken as the quadratic $v_o = 4H\xi(1 - \xi)$ so that the initial curvature is $v_{o,\xi\xi} = -8H$ which is constant. The quantity H is the maximum value of v_o (or the initial beam rise) and occurs at the beam center, $\xi = 1/2$. When the relations (5) are used in equation (4) and both sides of equation (4) are multiplied by appropriate factors, the following result is obtained:

$$(L^3/E_f I_f h^2)L_r = J, \tag{6}$$

where J is given by

$$J = \int_0^1 [V_{,\xi\xi}^2 + \alpha(U_{,\xi} + 8\Delta V)^2 + (1/\beta)(2U - V_{,\xi})^2]\, \mathrm{d}\xi$$

$$- \Omega^2 \int_0^1 [r_2 V^2 + rU^2]\, \mathrm{d}\xi, \tag{7}$$

where the following non-dimensional parameters have been defined: $\beta = 2E_f I_f / G_c A_c L^2$, transverse shear parameter; $\Omega^2 = \omega^2 m_f L^4 / E_f I_f$, frequency parameter; $\Delta = H/h$, initial rise parameter; the parameters $\alpha = A_f h^2 / I_f$, $r_2 = 1 + m_c/2m_f$, $r = h^2/L^2 + I_{mc}/2m_f L^2$ involve the geometry and density of the face sheets and core.

The stationary condition for J, namely $\delta J = 0$, will lead to the corresponding governing non-dimensional differential equations and boundary conditions. Accordingly, by taking the variation of J, integrating by parts, and using the usual variational calculus, the following differential equations and boundary conditions are obtained:

$$V_{,\xi\xi\xi\xi} - \frac{1}{\beta} V_{,\xi\xi} + \left(\frac{2}{\beta} + 8\alpha\Delta\right) U_{,\xi} + (64\alpha\Delta^2 - r_2\Omega^2) V = 0, \tag{8}$$

$$\alpha U_{,\xi\xi} + \left(\frac{2}{\beta} + 4\alpha\Delta\right) V_{,\xi} - \left(\frac{4}{\beta} - r\Omega^2\right) U = 0. \tag{9}$$

The corresponding boundary conditions are as follows. At $\xi = 0, 1$, we are to prescribe either

$$\alpha(U_{,\xi} + 8\Delta V) \quad \text{or} \quad U, \qquad V_{,\xi\xi} \quad \text{or} \quad V_{,\xi}, \tag{10, 11}$$

$$V_{,\xi\xi\xi} + \frac{1}{\beta}(2U - V_{,\xi}) \quad \text{or} \quad V. \tag{12}$$

The boundary conditions on the left of the "or" in equations (10)–(12) are the so-called natural boundary conditions, and are a consequence of the variational statement $\delta J = 0$. They need not be satisfied by an assumed series in a valid Rayleigh–Ritz approach using the functional in equation (7), although their satisfaction may improve the calculated results when finite series are used. The geometric boundary conditions, however, to the right of "or" in equations (10)–(12) must be satisfied by any assumed series.

It is noted that the structure of the derivatives in the differential equations (8) and (9) is such that expressions for V and U given by

$$V = a_n \sin(n\pi\xi), \qquad U = b_n \cos(n\pi\xi), \tag{13}$$

will satisfy them exactly. In other words there is decoupling for each n, and the natural frequencies for each mode number, n, can be calculated separately. In fact, if a series of the functions given in equation (13) is used in the Rayleigh–Ritz method using the functional J, orthogonality of these functions will again allow de-coupling of the modes. This will not be the case for other boundary conditions or when a delamination is present. Equations (13) satisfy boundary conditions which are termed roller simple supports, so that at $\xi = 0, 1$ $V = 0$, $V_{,\xi\xi} = 0$, and the vanishing of the axial load in terms of displacements gives $U_{,\xi} + 8\Delta V = 0$.

For the case of fixed simple supports, where $V = V_{,\xi\xi} = U = 0$ at $\xi = 0$ and 1, appropriate N term series for V and U, which satisfy these boundary conditions, can be taken as

$$(V, U) = \sum_{n=1}^{N} (A_n, B_n) \sin (n\pi\xi). \tag{14}$$

On substitution of the Fourier series (14) into equation (7) for no delamination, using orthogonality and other integral properties of the trigonometric functions, and minimizing J with respect to the coefficients in equation (14), i.e., $\partial J/\partial A_n = \partial J/\partial B_n = 0$, yields $2N$ simultaneous equations for the coefficients A_n, $B_n (n = 1, 2, \ldots, N)$. These can be written as

$$\left(n^4\pi^4 + 64\alpha\Delta^2 + \frac{n^2\pi^2}{\beta} - \lambda r_2\right)A_n + \sum_{\substack{k=1 \\ (n+k=\mathrm{odd})}}^{N} \left(32\alpha\Delta + \frac{8}{\beta}\right)\frac{nk}{n^2 - k^2} B_k = 0, \tag{15}$$

$$\sum_{\substack{k=1 \\ (n+k=\mathrm{odd})}}^{N} \left(32\alpha\Delta + \frac{8}{\beta}\right)\frac{nk}{k^2 - n^2} A_k + \left(n^2\pi^2\alpha + \frac{4}{\beta} - \lambda r\right)B_n = 0, \tag{16}$$

where the eigenvalue is denoted by $\lambda = \Omega^2$. In equations (15) and (16), n and k have values ranging from 1 to N, where the sums over k include only terms for which $n + k = \mathrm{odd}$. Note how the summation terms in equations (15) and (16) which involve the parameters Δ and β due to initial curvature and shear deformation respectively, lead to the coupling of the various coefficients in the series (14). A sufficient number of terms, N, are chosen to provide converged results.

For the case where a delamination is present between the core and one of the face sheets (see Figure 1), the functional, J, needs to be altered. In the delaminated region, there is no transverse shear strain energy in the core since the shear stress is zero at the delamination edge, and there is no mechanism to have it develop through the thickness since the core is assumed to have no axial stresses. This means that the strain energy due to transverse shear and the core rotary inertia term must be subtracted from the integral over the damaged region so that the functional with a delaminated region is written as

$$J_d = J - \int_{\xi_1}^{\xi_2} \frac{1}{\beta} (2U - V_{,\xi})^2 \, \mathrm{d}\xi + \lambda \int_{\xi_1}^{\xi_2} r_1 U^2 \, \mathrm{d}\xi, \tag{17}$$

where the core moment of inertia is contained in $r_1 = I_{mc}/2m_f L^2 = r - h^2/L^2$. In the calculations, a symmetric delamination is selected for convenience, as in Figure (1), so that the limits of integration in equation (17) are $\xi_1 = \frac{1}{2} - e$ and $\xi_2 = \frac{1}{2} + e$, where e is the half

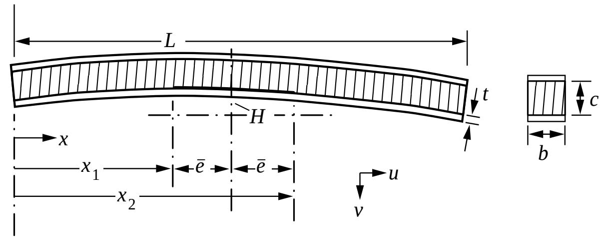

Figure 1. Geometry and notation.

length of delamination. Except for more algebra, there should be no difficulty in utilizing the analytical approach developed here to handle arbitrary delamination location. Minimizing J_d, as given by equation (17) leads to $2N$ simultaneous equations for A_n and B_n ($n = 1, 2, \ldots, N$), similar to equations (15) and (16), but now also containing additional coupling terms arising from the delamination half length, e. These equations are given in the Appendix A as equations (A.1) and (A.2). Standard procedures are used to determine the eigenvalues for the quantity, $\lambda = \Omega^2$.

3. RESULTS AND DISCUSSION

In the results to follow, consideration is given to the fundamental frequency of lateral vibration of the sandwich beam. This mode exhibited a rather interesting interaction of the effects of initial curvature, core shear deformation and the extent of delamination between the core and the face sheet. The frequency of longitudinal motion of the face sheets was not affected in any dramatic way (maximum variations of the order of 10% or less) by the values chosen for the parameters in the present study. In the calculations, typical values for the parameters given after equation (7), which involve geometry and mass, have been taken as $\alpha = 1000$, $r = 0{\cdot}01$ and $r_2 = 2$.

In Figure 2 the fundamental frequency, Ω, is plotted against the initial rise, Δ, for various values of the shear parameter, β. Note the substantial increase in frequency even for small initial curvature. For instance, for an initial rise, $\Delta = 0{\cdot}1$, and moderate shear deformation ($\beta = 0{\cdot}10$) a 90% increase in frequency is obtained. When there is less shear deformation ($\beta = 0{\cdot}01$), the corresponding increase in frequency is 22%. Now, the parameter, Δ, can be written as $\Delta = H/h = (H/t)(t/h)$. In the present calculations, t/h has the value $0{\cdot}11$, so that when $\Delta = 0{\cdot}1$ (as taken above), the quantity $H/t = 0{\cdot}91$. This means (see Figure 1) that the initial rise is not quite equal to the thickness of the face sheet. Also indicated in Figure 2 is the fact that when initial curvature increases (higher values of Δ), the influence of β becomes less. It can be seen from Figure 2 that when the beam is straight

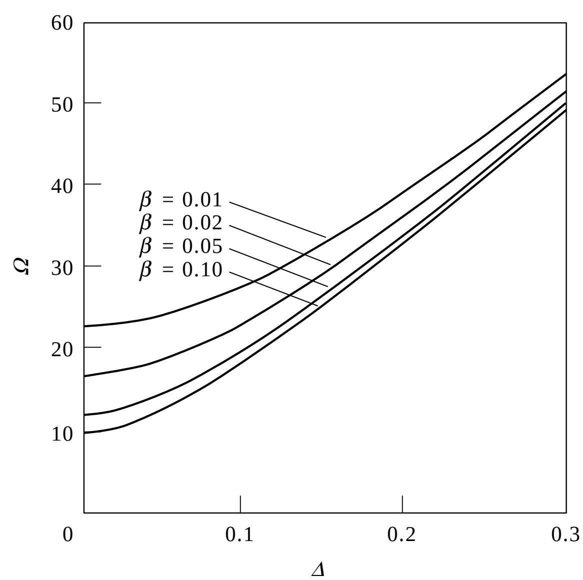

Figure 2. Fundamental frequency, Ω, versus initial rise, Δ, for various values of shear parameter β; $e = 0$.

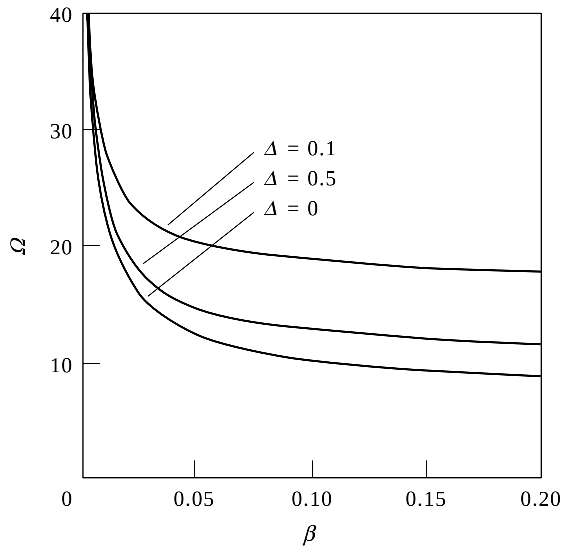

Figure 3. Fundamental frequency, Ω, versus β for various values of initial rise parameter, Δ; $e = 0$.

($\Delta = 0$), as β increases from 0·01 to 0·10 the frequency drops by about 45%. When the beam is initially curved so that $\Delta = 0{\cdot}3$, the corresponding drop is only about 7%.

The fundamental frequency is plotted against β in Figure 3 for various values of Δ. After a sharp drop with β initially, the curves flatten out beyond certain values of β (higher shear deformation). It is noted here that for the straight sandwich beam case ($\Delta = 0$), as β becomes very large (where the weak core begins to have less influence on the face sheet

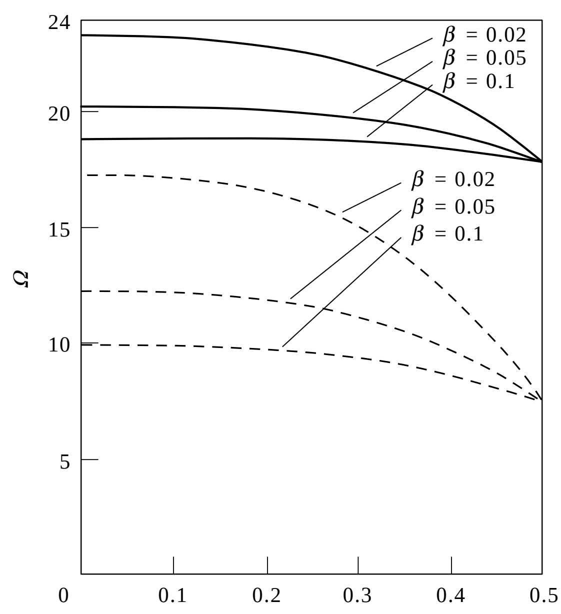

Figure 4. Fundamental frequency, Ω, versus e (half length of delamination) for various values of β for the straight (---, $\Delta = 0$) and initially curved (—, $\Delta = 0{\cdot}1$) beam.

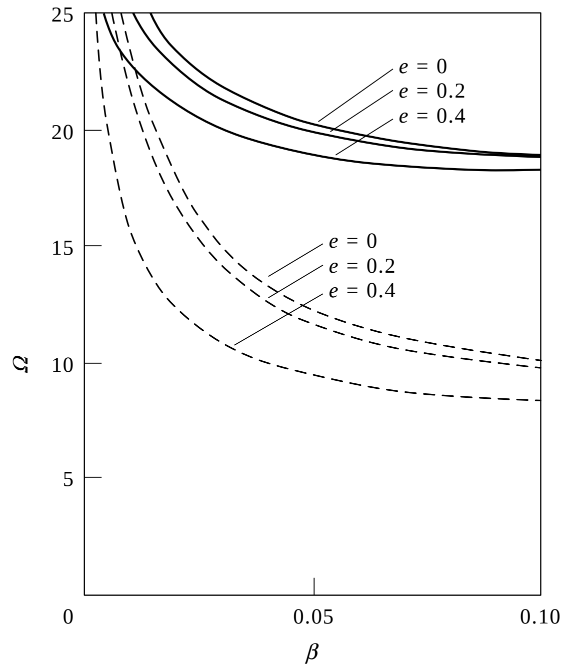

Figure 5. Fundamental frequency, Ω, versus β for various values of e for the straight (---, $\Delta = 0$) and initially curved (—, $\Delta = 0{\cdot}1$) beam.

deformation), there will be a decoupling of the lateral and longitudinal motion of the face plates. This is reflected by the decoupling of A_n and B_n in equations (15) and (16). As $\beta \to \infty$ (for $\Delta = 0$), setting the coefficient of $A_n = 0$ for $n = 1$, leads to $\lambda r_2 = \pi^4$, so that $\lambda = \Omega^2 = \pi^4/2$ or $\Omega = 6{\cdot}98$. The bottom curve in Figure 3 will become asymptotic to this value as $\beta \to \infty$.

In Figure 4 the fundamental frequency is plotted against the half length of delamination for various values of β for both the straight beam and initially curved beam ($\Delta = 0{\cdot}1$). Note how the sensitivity to the delamination parameter is greater for the straight beam than for the initially curved beam ($\Delta = 0{\cdot}1$). For instance, for moderate shear deformation ($\beta = 0{\cdot}02$), a 60% delamination length ($e = 0{\cdot}3$), causes a 15% drop in frequency for the straight beam compared to a 6% drop for the curved beam. The influence of the delamination is rather small for the higher values of β (high shear deformation). To state it differently, the influence of a delamination is small in an already weak core.

In Figure 5, Ω is plotted against β for various values of e for a straight and a curved beam. Once again we see the greater sensitivity of the straight beam to the delamination than that of the curved beam.

4. CONCLUSIONS

A non-dimensional variational statement is developed for a slightly curved beam, which may contain a delamination between the core and the face sheet. Non-dimensional parameters are defined which involve initial curvature (or beam rise), core shear deformation, core/face sheet geometry and delamination length. The numerical results for the fundamental frequency exhibit the interesting interplay of initial curvature, transverse shear deformation and delamination length. For instance, it is shown that when initial curvature increases, the influence of shear deformation becomes less. Also the sensitivity of the frequency to the delamination length is greater for straight beams than for initially curved beams. It is also greater for cases with less shear deformation (i.e., high modulus cores). In all cases, it is shown how even very slight curvature tends to increase the fundamental frequency sharply.

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APPENDIX: EQUATIONS FOR A_k AND B_k

$$\left[n^4\pi^4 + 64\alpha\Delta^2 + \frac{n^2\pi^2}{\beta}(1 - 2F_4(n, e)) - \lambda r_2\right]A_n - \frac{n\pi^2}{\beta}\sum_{\substack{k=1 \\ (n+k=\text{even})}}^{N} kF_3(n, k, e)A_k$$

$$+ \sum_{\substack{k=1 \\ (n+k=\text{odd})}}^{N} \frac{nk}{n^2 - k^2}\left(32\alpha\Delta + \frac{8}{\beta}\right)B_k + \frac{4n\pi}{\beta}\sum_{\substack{k=1 \\ (n+k=\text{odd})}}^{N} F_5(k, n, e)B_k = 0, \quad \text{(A.1)}$$

$$\sum_{\substack{k=1 \\ (k+n=\text{odd})}}^{N} \frac{nk}{k^2 - n^2}\left(32\alpha\Delta + \frac{8}{\beta}\right)A_k + \frac{4\pi}{\beta}\sum_{\substack{k=1 \\ (k+n=\text{odd})}}^{N} kF_5(n, k, e)A_k$$

$$+\left[n^2\pi^2\alpha + \frac{4}{\beta}(1 - 2F_2(n, e)) - \lambda(r - 2r_1F_2(n, e))\right]B_n - \frac{4}{\beta}\sum_{\substack{k=1 \\ (n+k=\text{even})}}^{N} F_1(k, n, e)B_k$$

$$-\lambda r_1 \sum_{\substack{k=1 \\ (n+k=\text{even})}}^{N} F_1(k, n, e)B_k = 0, \quad \text{(A.2)}$$

where

$$F_1(k, n, e) = \frac{(-1)^{(k-n)/2}}{\pi}(g_1 - g_2), \qquad F_2(n, e) = e - g_3,$$

$$F_3(k, n, e) = \frac{(-1)^{(k-n)/2}}{\pi}(g_1 + g_2), \qquad F_4 = e + g_3,$$

$$F_5(k, n, e) = \frac{\sin\left(\dfrac{k-n}{2}\,\pi\right)}{\pi}(g_1 + g_2),$$

$$g_1 = \frac{\sin[(k-n)\pi e]}{k-n}, \qquad g_2 = \frac{(-1)^n \sin[(k+n)\pi e]}{k+n}, \qquad g_3 = \frac{(-1)^n \sin(2n\pi e)}{2n\pi}.$$